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STABILITY AND SELF-OSCILLATIONS OF A ROTOR CONTAINING A CONDUCTING LIQUID IN A MAGNETIC FIELD

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This paper considers the problem of the stability in the small of the steady-state spinning of a rotor with a cylindrical cavity partly filled with a viscous, incompressible, conducting liquid in a magnetic field. The responses of the butt-end boundary layers and the resultant force exerted by the liquid on the rotor performing circular precession of small radius are determined. The plane of the viscoelastic restraint parameters of the rotor axis was D-partitioned into regions with different degrees of instability is constructed. Steady-state spinning near the boundary of the region of stability in the space of parameters is studied assuming nonlinear responses of the supports. It is shown that passage through the boundary of the region of stability leads to bifurcation of the steady-state spinning regime, resulting in periodic motion of the type of circular precession. The origin of periodic motion from steady-state spinning can be subcritical or supercritical.

Key words: *self-oscillations, magnetic field, stability of motion, conducting liquid.*

Introduction. Rotor systems containing liquids have found increasing applications, in particular, in the separation of soluble substances. Instabilities of the steady-state spinning of a rotor partly filled with a liquid are due, first, to resonance excitation of waves in the liquid [1–3]. Because the liquids in rotors commonly have conductivity, it seems possible to use a magnetic field to damp the wave resonances of the conducting liquid and, hence, to obtain an additional means for stabilizing steady-state rotor spinning.

The following remark needs to be made. Apparently, a homogeneous constant magnetic field directed along the rotor spinning axis is the easiest to implement. Laval restraint of the rotor axes is widely used; in this case, the angular displacements of the rotor axis are negligible. For restraints of this type, it is usually assumed that the particles of the liquid and the rotor move in planes perpendicular to the steady-state spinning axis. However, when the conducting liquid performs plane–parallel motion in a magnetic field collinear to the spinning axis, the resultant ponderomotive force is equal to zero and the magnetic field does not exert an influence on the stability of the rotor [4]. Hence, a mathematical model that includes the magnetic field effect should take into account that the liquid motion is not plane parallel, at least near the butt ends.

In the present study, the stability of the steady-state rotor spinning is examined using a version of the D-partitioning method [5] that does not employ a secular equation. The plane of the viscoelastic restraint parameters of the rotor axis is partitioned into regions with different degrees of instability.

1. Equations and Boundary Conditions. Let a rotor (Fig. 1) having a long cylindrical cavity be partly filled with a viscous, incompressible, conducting liquid. The external magnetic field is constant and homogeneous, and is directed along the spinning rotor axis. The points of the rotor can move only in planes perpendicular to the steady-state spinning axis. The absolute angular velocity of spinning is kept constant and equal to Ω.

We introduce a fixed coordinate system $O_1x_1x_2x_3$ with the axis O_1x_3 coincident with the steady-state spinning axis. The rotor axis is under axisymmetric viscoelastic restraint, which is generally speaking nonlinear. The geometry of the structure is such that is possible to neglect angular displacements of the rotor and assume that

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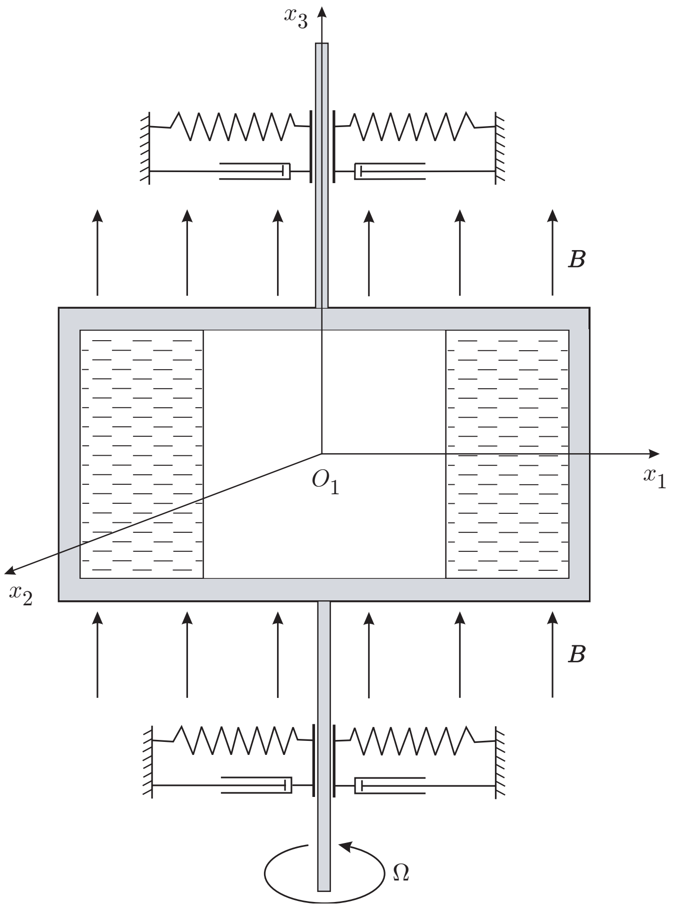

Fig. 1

the points of the rotor move in planes perpendicular to the steady-state spinning axis. We also ignore the elasticity of the nonconducting rotor walls.

The equations of motion of the rotor are written in complex variables:

$$M\ddot{z} = F + f, \qquad \dot{\theta} = \Omega,$$

$$z = x_1^0 + ix_2^0, \qquad F = F_1 + iF_2, \qquad i^2 = -1, \qquad F_k = -\iint\limits_S \sigma_{kj} n_j \, ds, \tag{1}$$

$$\sigma_{kj} = -p\delta_{kj} + \mu\Big(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_k}{\partial x_k}\Big), \qquad j = 1, 2, \quad k = 1, 2,$$

$$f = -(K + K_\alpha |z|^\alpha) z - (H + H_\beta |z|^\beta)\dot{z}.$$

Here $0 < \alpha \leqslant 1$ and $0 < \beta \leqslant 1$ are numerical parameters, M is the mass of the cylinder, x_1^0 and x_2^0 are the coordinates of the point of intersection of the rotor axis with the plane $O_1x_1x_2$, F_k are the hydrodynamic-force components, σ_{kj} are the stress-tensor components, μ is the dynamic viscosity of the liquid, f is the response of the rotor axis support, and K and H are linear elasticities and viscosities and K_α and H_β are the nonlinear elasticities and viscosities of the restraints of the rotor axis, respectively.

For the spinning rates and rotor dimensions used in engineering, one can assume that the magnetic Reynolds number is small and use a noninduction approximation. In view of this, we employ the following equations of motion of a viscous, incompressible, conducting liquid in the rotor cavity:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\,\nabla)\,\boldsymbol{v} = -\frac{\nabla p}{\rho} + \nu\,\Delta\boldsymbol{v} + \frac{\sigma}{\rho}(\boldsymbol{E} + [\boldsymbol{v}, \boldsymbol{B}], \boldsymbol{B}), \qquad \operatorname{div}\boldsymbol{v} = 0. \tag{2}$$

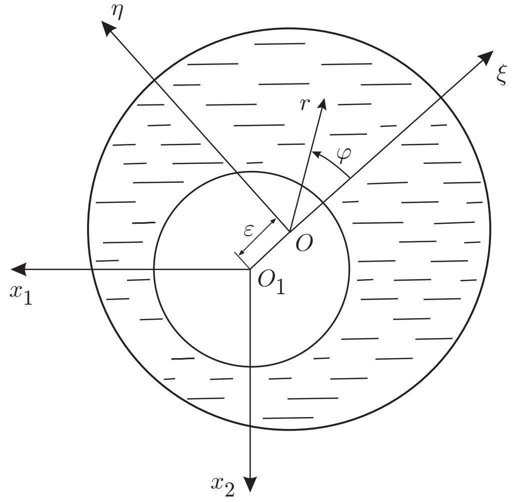

Fig. 2

Here ρ is the liquid density, $\nu = \mu/\rho$ is the kinematic viscosity, and σ is the conductivity. The equations for the electric field $\boldsymbol{E}$ are written as

$$\operatorname{rot} \boldsymbol{E} = 0, \qquad \operatorname{div} \boldsymbol{E} + (\boldsymbol{B} \operatorname{rot} \boldsymbol{v}) = 0. \tag{3}$$

The boundary conditions have the form

$$\boldsymbol{v}\Big|_s = \boldsymbol{v}_s, \qquad \frac{\partial \Phi}{\partial t} + (\boldsymbol{v}, \nabla\Phi)\Big|_{\Phi=0} = 0, \qquad \Phi(x_1, x_2, x_3, t) = 0,$$

$$(\boldsymbol{E}, \boldsymbol{n})\Big|_s = -([\boldsymbol{v}_s, \boldsymbol{B}], \boldsymbol{n}), \qquad (\boldsymbol{E} + [\boldsymbol{v}, \boldsymbol{B}], \nabla\Phi)\Big|_{\Phi=0} = 0, \tag{4}$$

$$\sigma_{kj} \frac{\partial \Phi}{\partial x_j}\Big|_{\Phi=0} = -p_* \frac{\partial \Phi}{\partial x_k}.$$

Here $\boldsymbol{n}$ is the normal to the rotor wall s, $\Phi(x_1, x_2, x_3, t) = 0$ is the equation of the free surface, $\boldsymbol{v}_s = (\dot{x}_1^0, \dot{x}_1^0, 0)$, and p_* is the pressure on the free surface.

2. Variation of the Stability and Circular Precession. In [2], a method is proposed to study, in a linear approximation, the stability of steady-state spinning for a rotor of cylindrical symmetry with a cavity partly filled with a viscous incompressible liquid for the case where the angular velocity of the rotor is kept constant and its axis is under axisymmetric viscoelastic restraint. This stability analysis method is generalized in [3]. After linearization near the steady-state spinning regime, the equations of motion (1)–(4) and the boundary conditions admit solutions proportional to $\exp(\lambda t)$ (λ is a characteristic number). The steady-state spinning regime is stable if all values of λ have negative real parts, and it is unstable if even one of the values of λ has a positive real part. For a continuous dependence of λ on the problem parameters, the degree of instability changes when a pair of characteristic numbers passes through the imaginary axis. Following [2], one can show that imaginary characteristic numbers exist if and only if the system admits a solution of the type of circular precession. Hence, the parameter value for which the degree of instability of the system changes can be found from the condition for the existence of circular precession. Thus, one first needs to consider the magnetohydrodynamic problem of the motion of the conducting liquid for the case of circular precession of the rotor and to find the force exerted by the liquid on the rotating cylinder. Next, using the expressions obtained for the force and the linearized equations of translation motion of the rotor (1), one obtains conditions under which the circular precession is possible. These conditions determine the boundaries of the regions with different degrees of instability in the space of the problem parameters.

3. Plane Problem and Boundary-Layer Problem. Let the rotor performs circular precession of small radius ε with frequency ω. We introduce a moving coordinate system $O\xi\eta\zeta$ so that the coordinate origin coincides with the center of the cross-section of the cylindrical cavity of the rotor O and the axis $O\xi$ is directed along the line connecting the center of the precession O_1 and the point O (Fig. 2). The axis $O\zeta$ is parallel to the axis Ox_3. We also introduce a cylindrical coordinate system $r\varphi\zeta$ related to the coordinate system $O\xi\eta\zeta$. In the steady-state spinning regime, the liquid rotates together with the rotor as a unit:

$$\boldsymbol{v} = \boldsymbol{v}_0 \equiv \omega_0 r \boldsymbol{e}_\varphi, \qquad p = p_0 \equiv \rho\Omega(r^2 - b^2)/2 + p_*, \qquad \boldsymbol{E} = \boldsymbol{E}_0 \equiv -[\boldsymbol{v}_0, \boldsymbol{B}]$$

($\omega_0 = \Omega - \omega$).

In the case of circular precession, the quantities $\boldsymbol{v}$, p, Φ, and $\boldsymbol{E}$ are expanded in powers of the small parameter ε near the steady-state spinning regime:

$$\boldsymbol{v} = \boldsymbol{v}_0 + \varepsilon\boldsymbol{v}_1 + \varepsilon^2\boldsymbol{v}_2 + \ldots, \qquad p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots,$$

$$\boldsymbol{E} = \boldsymbol{E}_0 + \varepsilon\boldsymbol{E}_1 + \varepsilon^2\boldsymbol{E}_2 + \ldots, \qquad \Phi = r - b - \varepsilon h(\varphi) + \ldots \,.$$

In the chosen noninertial coordinate system $O\xi\eta\zeta$, the liquid motion during the circular precession is described by time-independent functions [6], and, in the first order in ε, it is defined by the following linearized equations of magnetohydrodynamics:

$$\begin{gathered}\operatorname{rot}[\boldsymbol{v}_1, \boldsymbol{v}_0] = -2[\Omega, \boldsymbol{v}_1] - \frac{\nabla p_1}{\rho} + \omega^2\boldsymbol{e}_\xi + \frac{\sigma}{\rho}[\boldsymbol{E}_1, \boldsymbol{B}] - \frac{\sigma B^2}{\rho}\boldsymbol{v}_1 + \nu\,\Delta\boldsymbol{v}_1 + \frac{\sigma[\boldsymbol{B}, \boldsymbol{v}_1]}{\rho}\boldsymbol{B}, \\ \operatorname{div}\boldsymbol{v}_1 = 0, \qquad \operatorname{rot}\boldsymbol{E}_1 = 0, \qquad \operatorname{div}\boldsymbol{E}_1 = -(\boldsymbol{B}, \operatorname{rot}\boldsymbol{v}_1).\end{gathered} \tag{5}$$

In this case, the solution of system (5) with the corresponding boundary conditions can be written as the sum of two terms: the solutions of the problem of plane parallel liquid motion $\boldsymbol{v}_1^p$, p_1^p, and $\boldsymbol{E}_1^p$ and the solutions of the boundary-layer problem $\boldsymbol{v}_1^b$, p_1^b, and $\boldsymbol{E}_1^b$ taking into account the effect of the cylindrical cavity near the butt ends. In other words, it can be assumed that except in small regions near the butt ends, the particles of the liquid move in planes parallel to the plane $O\xi\eta$. This motion is described by Eqs. (5) in which the velocity component along the axis $O\zeta$ and the derivatives with respect to ζ are set equal to zero. In boundary conditions (4), one should retain only the condition of attachment of the liquid, the condition of nonpenetration of electric current through the vertical walls of the cylinder, and the kinematic and dynamic conditions on the free cylindrical surface of the liquid:

$$\boldsymbol{v}_1^p\Big|_{r=a} = 0, \qquad \omega_0\frac{\partial h}{\partial\varphi}\Big|_{r=b} = u_1^p\Big|_{r=b},$$

$$E_{1r}^p\Big|_{r=a} = 0, \qquad E_{1r}^p + v_1^p B\Big|_{r=b} = 0,$$

$$-p_1^p - \rho\Omega^2 rh + 2\mu\frac{\partial u_1^p}{\partial r}\Big|_{r=b} = 0, \qquad \frac{\partial v_1^p}{\partial r} + \frac{1}{r}\frac{\partial u_1^p}{\partial\varphi} - \frac{v_1^p}{r}\Big|_{r=b} = 0.$$

Here u_1^p and v_1^p are the radial and azimuthal velocity components $\boldsymbol{v}_1^p = (u_1^p, v_1^p, 0)$, $r = b + \varepsilon h(\varphi)$ (free-surface equation), and E_{1r}^p is the component of the vector $\boldsymbol{E}_1^p$.

The solution of the purely hydrodynamic plane problem is found in [2]. It is easy to show that the magnetic field does not influence the velocity field $\boldsymbol{v}_1^p$ and the solution can be written as

$$\boldsymbol{E}_1^p = -[\boldsymbol{v}_1^p, \boldsymbol{B}],$$

$$u_1^p = \Big[c_1 + \frac{c_2}{r^2} + \frac{i}{r}Z_1(kr)\Big]\mathrm{e}^{i\varphi} + \text{c.c.}, \qquad v_1^p = \Big[ic_1 - \frac{ic_2}{r^2} - kZ_0(kr) + \frac{1}{r}Z_1(kr)\Big]\mathrm{e}^{i\varphi} + \text{c.c.},$$

$$\frac{p_1^p}{\rho} = \Big[i(2\Omega - \omega_0)c_1 r + i(2\Omega + \omega)\frac{c_2}{r} + \frac{\omega^2 r}{2} - 2\Omega Z_1(kr)\Big]\mathrm{e}^{i\varphi} + \text{c.c.},$$

where $Z_n(kr) = c_3 H_n^{(2)}(kr) + c_4 H_n^{(1)}(kr)$, $H_n^{(1),(2)}(kr)$ is a Hankel function of the nth order, $k = x(i - \omega_0/|\omega_0|)$, $x = \sqrt{|\omega_0|/(2\nu)}$, and c.c. denotes the expression which is complex conjugate to the previous expression.

Viscous boundary layers of thickness $\sqrt{\nu/\Omega}$ form near the butt ends. Using boundary layer theory, we derive the system of equations describing liquid motion near the butt ends. We note that in the end boundary layers, the components of $\operatorname{rot}\boldsymbol{v}_1^b$ have different order: the rotor component normal to the butt end of the cylinder is much smaller than the other two. This implies that $\operatorname{div}\boldsymbol{E}_1^b$ can be approximately considered equal to zero. Then, from the homogeneous boundary conditions and the condition rot $\boldsymbol{E}_1^b = 0$, it follows that $\boldsymbol{E}_1^b$ is equal to zero at any point. In this case, for u_1^b and v_1^b, we have the equations

$$\omega_0 \frac{\partial u_1^b}{\partial \varphi} = 2\Omega v_1^b + \nu \frac{\partial^2 u_1}{\partial \zeta^2} - \frac{\sigma B^2}{\rho} u_1^b, \qquad \omega_0 \frac{\partial v_1^b}{\partial \varphi} = -2\Omega u_1^b + \nu \frac{\partial^2 u_1^b}{\partial \zeta^2} - \frac{\sigma B^2}{\rho} v_1^b \tag{6}$$

with the boundary conditions

$$u_1^b\Big|_{\zeta=0} = -u_1^p, \qquad v_1^b\Big|_{\zeta=0} = -v_1^p, \qquad u_1^b\Big|_{\zeta\to\infty} = 0, \qquad v_1^b\Big|_{\zeta\to\infty} = 0. \tag{7}$$

Here the conditions only for the lower butt ($\zeta = 0$) are given; the conditions at the other butt end are written similarly. The right sides of boundary conditions (7) include only the first harmonic in φ; the other harmonics are absent. From this it follows, that the solution of problem (6), (7) can be written as

$$u_1^b = \hat{u}(r,\zeta)\, \mathrm{e}^{i\varphi} + \text{c.c.}, \qquad v_1^b = \hat{v}(r,\zeta)\, \mathrm{e}^{i\varphi} + \text{c.c.}$$

Immediate substitution shows that Eqs. (6) are satisfied for the functions

$$\hat{u}(r,\zeta) = A_1(r)\, \mathrm{e}^{-\lambda_1 \eta} + A_2(r)\, \mathrm{e}^{-\lambda_2 \eta} + A_3(r)\, \mathrm{e}^{\lambda_1 \eta} + A_4(r)\, \mathrm{e}^{\lambda_2 \eta},$$

$$\hat{v}(r,\zeta) = -iA_1(r)\, \mathrm{e}^{-\lambda_1 \eta} + iA_2(r)\, \mathrm{e}^{-\lambda_2 \eta} - iA_3(r)\, \mathrm{e}^{\lambda_1 \eta} + iA_4(r)\, \mathrm{e}^{\lambda_2 \eta},$$

where

$$\lambda_n = \Big(\frac{\sigma^2 B^4}{\rho^2 \nu^2} + \frac{(\omega_0 + 2\Omega(-1)^{n+1})^2}{\nu^2}\Big)^{1/4} \mathrm{e}^{i\Theta_n} = \frac{1}{a}\, (\mathrm{Ha}^4 + (1 - \tau + 2(-1)^{n+1})^2 E_*^{-2})^{1/4}\, \mathrm{e}^{i\Theta_n},$$

$$\Theta_n = \frac{1}{2} \arctan \Big[\frac{\rho(\omega_0 + 2\Omega(-1)^{n+1})}{\sigma B^2}\Big] = \frac{1}{2} \arctan \Big[\frac{1 - \tau + 2(-1)^{n+1}}{\mathrm{Ha}^2\, E}\Big], \qquad n = 1, 2,$$

$\mathrm{Ha} = Ba\sqrt{\sigma/(\rho\nu)}$ is the dimensionless Hartman number, which characterizes the relation between magnetic and viscous forces, $E_* = \nu/(\Omega a)^2$ is the dimensionless Ekman number, and $\tau = \omega/\Omega$.

From the boundary conditions, one obtains the functions $A_n(r)$:

$$A_1 = -\Big[\frac{c_2}{r^2} + \frac{i}{r}\, Z_1(kr) - \frac{ik}{2}\, Z_0(kr)\Big], \qquad A_2 = -\Big[c_1 + \frac{ik}{2}\, Z_0(kr)\Big], \qquad A_3 = A_4 = 0.$$

The corresponding solution for the boundary layer at the upper butt end is similar. The coefficients c_n are determined from the boundary conditions written as follows:

$$c_1 + \frac{c_2}{a^2} + i\, \frac{c_3}{a} H_1^{(2)}(ka) + i\, \frac{c_4}{a}\, H_1^{(1)}(ka) = 0,$$

$$c_1 - \frac{c_2}{a^2} + i\, \frac{c_3}{a}[kaH_0^{(2)}(ka) - H_1^{(2)}(ka)] + i\, \frac{c_4}{a}\, [kaH_0^{(1)}(ka) - H_1^{(1)}(ka)] = 0,$$

$$4\, \frac{c_2}{b^2} - i\, \frac{c_3}{a} \Big[2kaH_0^{(2)}(kb) + (k^2b^2 - 4)\, \frac{a}{b}\, H_1^{(2)}(kb)\Big] - i\, \frac{c_4}{a} \Big[2kaH_0^{(1)}(kb) + (k^2b^2 - 4)\, \frac{a}{b}\, H_1^{(1)}(kb)\Big] = 0,$$

$$-\frac{i\tau^2}{1-\tau}\, c_1 + i\Big[\frac{2 - 4\tau + \tau^2}{1-\tau} - \frac{4(1-\tau)}{k^2b^2}\Big]\frac{c_2}{b^2} + \Big[-\frac{2(1-\tau)}{kb}\, H_0^{(2)}(kb) + \Big(\frac{2\tau - 1}{1-\tau} + \frac{4(1-\tau)}{k^2b^2}\Big) H_1^{(2)}(kb)\Big] b^{-1} c_3$$
$$+ \Big[-\frac{2(1-\tau)}{kb}\, H_0^{(1)}(kb) + \Big(\frac{2\tau - 1}{1-\tau} + \frac{4(1-\tau)}{k^2b^2}\Big)\, H_1^{(1)}(kb)\Big] b^{-1} c_4 = -\frac{1}{2}\, \tau^2 \Omega. \tag{8}$$

The algebraic system of linear equations for c_n (8) is ill-conditioned; therefore, we convert to the quantities $\bar{c}_1 = c_1\Omega^{-1}$, $\bar{c}_2 = c_2 a^{-2}\Omega^{-1}$, $\bar{c}_3 = c_3 a^{-1}\Omega^{-1} H_0^{(2)}(ka)$, and $\bar{c}_4 = c_4 a^{-1}\Omega^{-1} H_0^{(1)}(kb)$. Below in this section, the bar above the symbols c_n is omitted. System (8) becomes

$$c_1 + c_2 + ic_3 h_{21} + ic_4 g_{20} = 0,$$

$$c_1 - c_2 + ic_3(ka - h_{21}) + ic_4(kag_{10} - g_{20}) = 0,$$

$$4c_2\delta^{-2} - ic_3[2kbh_{01} + (k^2b^2 - 4)\, \delta^{-1} h_{11}] - ic_4[2kb + (k^2b^2 - 4)\delta^{-1} g_{11}] = 0,$$

$$-\frac{i\tau^2}{1-\tau}\,c_1+i\Big[\frac{2-4\tau+\tau^2}{1-\tau}-\frac{4(1-\tau)}{k^2b^2}\Big]c_2\delta^{-2}+\Big[-\frac{2(1-\tau)}{kb}\,h_{01}+\Big(\frac{2\tau-1}{1-\tau}+\frac{4(1-\tau)}{k^2b^2}\Big)h_{11}\Big]\delta^{-1}c_3$$

$$+\Big[-\frac{2(1-\tau)}{kb}+\Big(\frac{2\tau-1}{1-\tau}+\frac{4(1-\tau)}{k^2b^2}\Big)g_{11}\Big]\delta^{-1}c_4=-\frac{1}{2}\,\tau^2.$$

Here

$$h_{01}=H_0^{(2)}(kb)/H_0^{(2)}(ka),\qquad h_{21}=H_1^{(2)}(ka)/H_0^{(2)}(ka),\qquad h_{11}=H_1^{(2)}(kb)/H_0^{(2)}(ka),$$

$$g_{20}=H_1^{(1)}(ka)/H_0^{(1)}(kb),\qquad g_{10}=H_0^{(1)}(ka)/H_0^{(1)}(kb),\qquad g_{11}=H_1^{(1)}(kb)/H_0^{(1)}(kb),$$

$\delta=b/a$ is the ratio of the thickness of the liquid layer adjacent to the wall for steady-state spinning to the radius of the cavity.

4. Hydrodynamic Forces. The force exerted by the liquid on the upper butt end of the cylinder is given by

$$F_\xi^b=-\int\limits_0^{2\pi}\int\limits_b^a(\sigma_{\zeta r}\cos\varphi-\sigma_{\zeta\varphi}\sin\varphi)\,r\,d\varphi\,dr=-2\pi\mu\,\mathrm{Re}\Big\{\lambda_2[c_1(a^2-b^2)+iaZ_1(ka)-ibZ_1(kb)]\Big\},$$

$$F_\eta^b=-\int\limits_0^{2\pi}\int\limits_b^a(\sigma_{\zeta r}\sin\varphi+\sigma_{\zeta\varphi}\cos\varphi)\,r\,d\varphi\,dr=-2\pi\mu\,\mathrm{Im}\Big\{\lambda_2[c_1(a^2-b^2)+iaZ_1(ka)-ibZ_1(kb)]\Big\}.$$

We calculate the hydrodynamic force acting on the side wall of the cylindrical cavity:

$$F_\xi^p=-\int\limits_0^{d}\int\limits_0^{2\pi}(\sigma_{rr}\cos\varphi-\sigma_{r\varphi}\sin\varphi)a\,d\varphi\,dz,\qquad F_\eta^p=-\int\limits_0^{d}\int\limits_0^{2\pi}(\sigma_{rr}\sin\varphi+\sigma_{r\varphi}\cos\varphi)a\,d\varphi\,dz$$

(d is the height of the cylinder). By virtue of the boundary conditions $u=0$ and $v=0$ on the wall $r=a$ and the incompressibility condition $\partial u/\partial r=0$ for $r=a$, the stress relations

$$\sigma_{rr}=-p+2\mu\,\frac{\partial u}{\partial r},\qquad \sigma_{r\varphi}=\mu\Big(\frac{1}{r}\,\frac{\partial u}{\partial\varphi}+\frac{\partial v}{\partial r}-\frac{v}{r}\Big)$$

are simplified:

$$\sigma_{rr}=-p,\qquad \sigma_{r\varphi}=\mu\,\frac{\partial v}{\partial r}.$$

After some transformations, we obtain

$$F_\xi^p=2\pi a^2\rho\varepsilon\,\mathrm{Re}\,[\omega^2/2+2i(\Omega-\omega)c_2],\qquad F_\eta^p=-4\pi a^2\rho\varepsilon(\Omega-\omega)\,\mathrm{Re}\,c_2.$$

The dependence of the total hydrodynamic force with the components $F_\xi=F_\xi^p+2F_\xi^b$ and $F_\eta=F_\eta^p+2F_\eta^b$ on the dimensionless frequency τ has a distinct resonance nature due to wave generation in the liquid.

5. Constructing Regions with Different Degrees of Instability. Using the method proposed in [4] to construct regions with different degrees of instability of the steady-state spinning regime in the plane of the restraint parameters of the cylinder axis, we find the values of the parameters K and H for which circular precession is possible. For this, we substitute the expression for the total hydrodynamic force into the equation of motion of the cylinder (1). After division by $m\varepsilon\Omega^2$ taking into account

$$F_1=\mathrm{Re}\,[(F_\xi+iF_\eta)\,\mathrm{e}^{i\omega t}],\qquad F_2=\mathrm{Im}\,[(F_\xi+iF_\eta)\,\mathrm{e}^{i\omega t}],\tag{9}$$

we obtain the relations linking ω and the problem parameters in the case of circular precession:

$$-M\tau^2/m+\bar K=\bar F_\xi\tau^2,\qquad \bar H\tau=\bar F_\eta\tau^2.\tag{10}$$

Here $m=\pi\rho(a^2-b^2)d$ is the mass of the liquid which partly fills the cavity of the rotor, $\bar K=K/(m\Omega)$, $\bar H=H/(m\Omega)$, and $\bar F_\xi=F_\xi/(m\omega^2)$ and $\bar F_\eta=F_\eta/(m\omega^2)$ are the dimensionless force components, which can be written as

$$\bar F_\xi=\frac{1}{1-\delta^2}\,\mathrm{Re}\Big\{1+4i\,\frac{1-\tau}{\tau^2}\,\bar c_2-\frac{4E_*\lambda_2 a}{\tau^2\bar\omega}\,[(1-\delta^2)\bar c_1+ic_3(h_{21}-\delta h_{11})+ic_4(g_{20}-\delta g_{11})]\Big\},$$

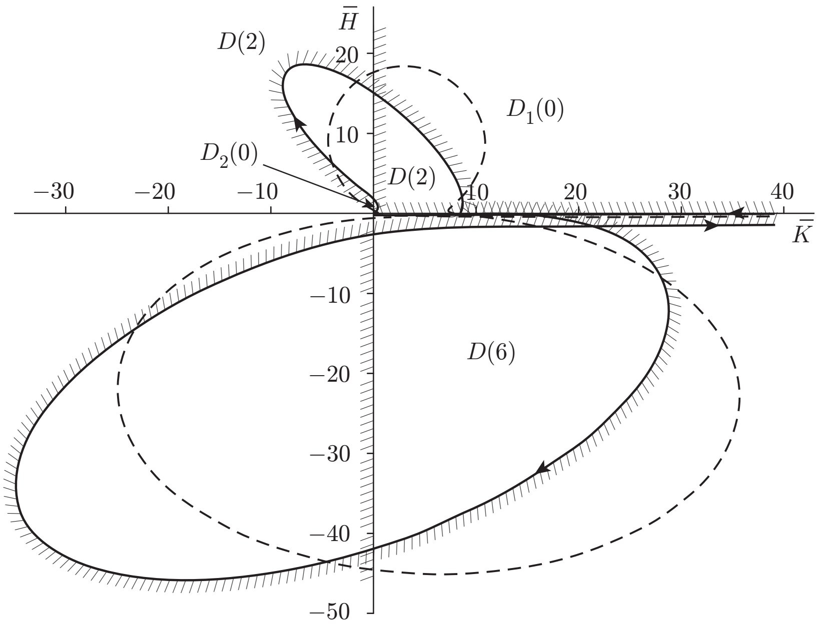

Fig. 3

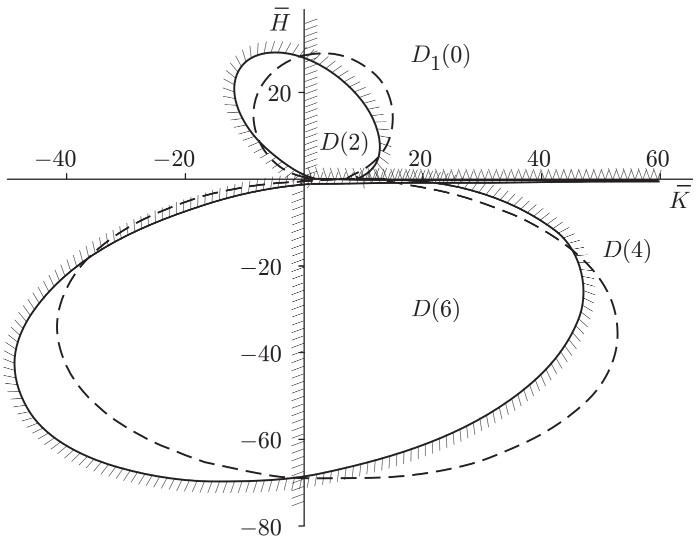

Fig. 4

$$\bar{F}_\eta = \frac{4}{1-\delta^2}\,\frac{1}{\tau^2}\,\mathrm{Im}\left\{-i(1-\tau)\bar{c}_2 - \frac{4E_*\lambda_2 a}{\bar{\omega}}\left[(1-\delta^2)\bar{c}_1 + ic_3(h_{21}-\delta h_{11}) + ic_4(g_{20}-\delta g_{11})\right]\right\}$$

($\bar{\omega} = d/a$ is the ratio of the height of the rotor cavity to its radius).

Figures 3 and 4 show the D-partitions of the plane of the parameters $\bar{K} = K/(m\Omega)$ and $\bar{H} = H/(m\Omega)$ into regions with different degrees of instability $D(n)$ (n is the degree of instability) for $\mathrm{Ha} = 1.4 \cdot 10^4$ and $E_* = 10^{-4}$, $5 \cdot 10^{-5}$, respectively. In both cases, the remaining parameters are identical: $M/m = 1.4$, $\delta = 0.92$, and $\bar{\omega} = 2$. The construction of the bifurcation curves (so-called D-curves) dividing the plane $(\bar{K}, \bar{H})$ into regions with different degrees of instability is described in detail in [2]. The hatching on the D-curves is conventional: transition from the hatched side of a curve onto the unhatched side of the curve in the plane of the parameters corresponds to an increase in the degree of instability by two unities. The arrow indicates the increasing direction of the parameter τ. For comparison, the D-partitions for the cases of no magnetic field ($\mathrm{Ha} = 0$) are shown by dashed curves. On the

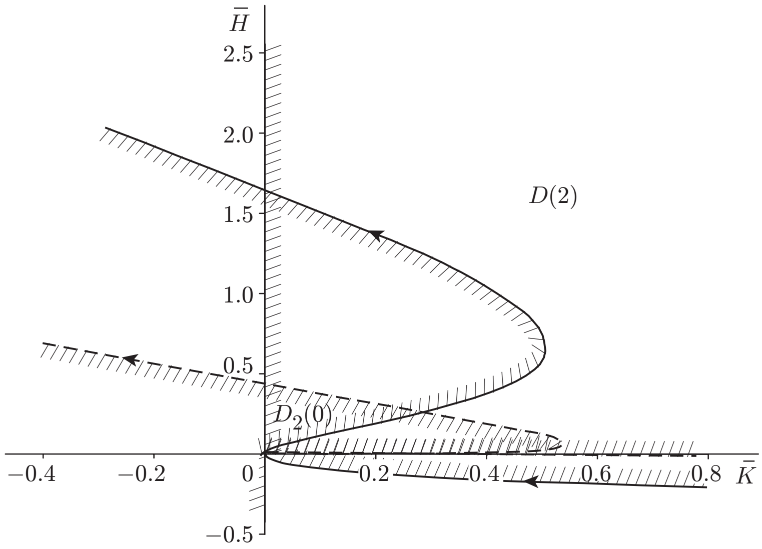

Fig. 5

dashed curves, there is no hatching. Apart from the region of stability $D_1(0)$ corresponding to large values of the damping and stiffness coefficients, there is a region of stability $D_2(0)$ near zero values of $\bar{K}$ and $\bar{H}$. This region is of the greatest interest from a practical view point. The region $D_2(0)$ (see Fig. 3) is scaled up in Fig. 5 ($E_* = 10^{-4}$, $M/m = 1.4$, $\delta = 0.92$, and $\bar{\omega} = 2$). From Figs. 3–5, it follows that rather strong magnetic fields lead to variations in the shape of the D-curve, among which we note a noticeable expansion of the region of stability $D_2(0)$ along the ordinate (the axis $\bar{H}$) with a slight contraction along the abscissa (the axis $\bar{K}$) and deformation with a left-hand shift of the entire curve of D.

The resultant of the ponderomotive forces arises in the butt-end layers; therefore, variation in E_* (the layer thickness is approximately equal to $aE_*^{1/2}$) leads to variation in the magnetohydrodynamic forces acting on the rotor. An increase in the Ekman number E_* diminishes the effect of the magnetic field (compare Fig. 4 and Fig. 5).

6. Behavior of the Steady-State Spinning Regime near the Boundary of the Region of Stability. Small variations of the parameters can result in the rotor system entering the boundary of the region of stability. Furthermore, in the case of an arbitrarily small violation of the "dangerous" boundary (see [7]), the system will enter a new state, which cannot be brought close to the initial state by a small violation of the boundary. Investigation of the nature of the boundaries is very important in stability analysis. Let us show that in the examined rotor model with a constant angular velocity and a nonlinear viscoelastic restraint of the axis, exit from the region of stability gives rise to Andronov–Hopf bifurcation (see [8–11]).

To elucidate whether passage through the boundary can lead to Andronov–Hopf bifurcation (i.e., the origin of periodic motion from the steady-state spinning regime), it is necessary to retain the principal nonlinear terms in the equations. We shall confine ourselves to the case where $\alpha = \beta = 1$ in the expression for the response of the supports. We seek a periodic solution in the form of circular precession:

$$z = \varepsilon \exp(i\omega t). \tag{11}$$

The right sides of the magnetohydrodynamic equations of the second approximation in ε contain the square of the first harmonic in φ, and the boundary conditions are zeroth. This implies that the solution of the second-approximation equations contains only the second or (and) zero harmonics in φ. Hence, the expansion of the hydrodynamic force $F_i = \iint \sigma_{ij} n_j \, ds$ in powers of ε does not contain a term with ε^2. Therefore, in considering the nonlinear problem, one can take into account only the nonlinear response of the restraint without determining the solution of the magnetohydrodynamic problem of the second approximation. Substituting (11) and F_ξ and F_η into (1), we obtain the following existence conditions for the periodic solution:

$$-M\omega^2 + K + K_1\varepsilon = F_\xi/\varepsilon, \qquad (H + H_1|\omega|\varepsilon)\omega = F_\eta/\varepsilon. \tag{12}$$

For $\varepsilon = 0$, the system of the final equations (12) for ε coincides with Eqs. (10), from which the boundaries of the regions with different degrees of instability were determined. Thus, for the parameter values belonging to the

D-curve, system (10) has a real solution with $\varepsilon = 0$ and $\omega \neq 0$. If this solution is a simple root of Eqs. (10), then for arbitrarily small deviations from the critical parameter values, by virtue of continuity, one obtains a close solution with real $\varepsilon \neq 0$ and $\omega \neq 0$. To occurrence of a solution with $\varepsilon > 0$ corresponds to the origin of periodic motion (in the form of circular precession) from the steady-state spinning regime ($\varepsilon = 0$), i.e., Andronov–Hopf bifurcation. Thus, we seek a solution of system (12) near the boundaries of the region $D_2(0)$ using the perturbation method, which leads to the following system of equations for ε and $\delta\tau$:

$$-\Big(2\,\frac{M}{m} + 2\bar{F}_\xi + \frac{\partial \bar{F}_\xi}{\partial \tau}\,\tau\Big)\tau\,\delta\tau + \bar{K}_1\varepsilon = -\delta\bar{K}, \qquad -\Big(-\bar{F}_\eta + \frac{\partial \bar{F}_\eta}{\partial \tau}\,\tau\Big)\delta\tau + \bar{H}_1|\tau|\varepsilon = -\delta\bar{H}$$

[$\bar{K}_1 = K_1 a/(m\Omega^2)$ and $\bar{H}_1 = H_1 a/m$].

Small increments of $\delta\bar{K}$ and $\delta\bar{H}$ are chosen so that the point $(\bar{K} + \delta\bar{K}, \bar{H} + \delta\bar{H})$ lies on the normal to the boundary issuing from the point $(\bar{K}, \bar{H})$. If periodic motion originates when the parameters leaves go out from the region of stability through any segment of its boundary, such bifurcation is called supercritical. In passage through such a segment, the excitation of self-oscillations proceeds smoothly, and, consequently, such segments of the boundary are commonly called "safe." In the neighborhood of "dangerous" segments, periodic motion in the form of circular precession of small radius exists in the region of stability of the steady-state spinning regime. This implies that in approaching such a segment from the region of stability, the system becomes unstable against perturbations of small but finite magnitudes and sever excitation of self-oscillations occurs.

The nature of the boundaries is significantly affected by the form of the elastic nonlinearity, i.e., the sign of the coefficient $\bar{K}_1$. In the case of "progressive" stiffness $\bar{K}_1 > 0$ and viscosity $\bar{H}_1 > 0$ for the above-mentioned values of the remaining parameters, the boundary of the region $D_2(0)$ is "dangerous." (In the calculations, the values of $\bar{K}_1$ were varied from 0.01 to 1 and the values of $\bar{H}_1$, from 10^{-3} to 1.) For "regressive" stiffnesses $\bar{K}_1 < 0$ ($\bar{H}_1 > 0$), the boundary of the region of stability $D_2(0)$ is "safe." The magnetic field has little effect on the nature of the boundaries.

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